A Transmission Problem in Electromagnetism with a Singular Interface

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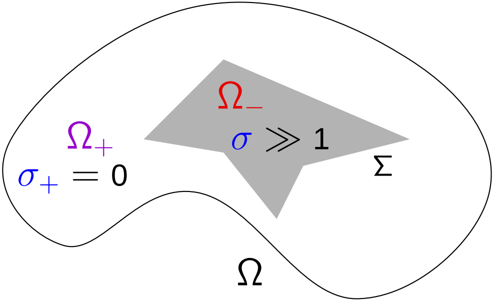
The Skin Effect : A Model Problem



- Ω_- Highly Conducting body $\subset\subset \Omega$: Conductivity $\sigma_- \equiv \sigma \gg 1$
- $\Sigma = \partial\Omega_-$: Interface
- Ω_+ Insulating or Dielectric body: Conductivity $\sigma_+ = 0$

The Skin Effect : rapid decay of electromagnetic fields with depth inside the conductor.

The Skin Depth : $\ell(\sigma) = \sqrt{2/\omega\mu_0\sigma}$

References

- V. PÉRON (PhD 09)
- G. CALOZ, M. DAUGE, V. PÉRON (JMAA 10)
  Uniform Estimates for Transmission Problems with High Contrast in Heat Conduction and Electromagnetism
- M. DAUGE, E. FAOU, V. PÉRON (CRAS 10)
  Asymptotic Behavior for High Conductivity of the Skin Depth Electromagnetism

- Aim : Understanding the influence of the geometry of a conducting body on the skin effect in electromagnetism.

Outline

Outline

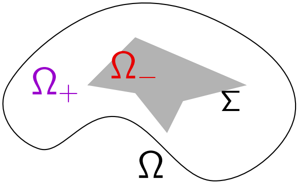
Framework

Maxwell Problem

$$(\mathbf{P}_{\underline{\sigma}}) \qquad \operatorname{curl}\mathbf{E} - i\omega\mu_0\mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl}\mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma})\mathbf{E} = \mathbf{j}$$

$$\underline{\sigma} = (0, \sigma \gg 1)$$

$$\mathbf{j} \in \mathrm{H}_0(\operatorname{div}, \Omega) = \{\mathbf{u} \in \mathrm{L}^2(\Omega) \mid \quad \operatorname{div}\mathbf{u} \in \mathrm{L}^2(\Omega),\ \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\}$$



Perfectly Conducting Magnetic Wall B. C.:

$$\mathbf{E}\cdot\mathbf{n} = 0 \quad \text{and} \quad \mathbf{H}\times\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega$$

Existence of solutions

Hypothesis (*SH*)

The angular frequency ω is not an eigenfrequency of the problem

$$\begin{cases} \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \quad \textit{and} \quad \operatorname{curl} \mathbf{H} + i\omega\varepsilon_0 \mathbf{E} = 0 & \textit{in} \quad \Omega_+ \\ \mathbf{E} \times \mathbf{n} = 0 & \textit{on} \quad \Sigma \\ B.C. & \textit{on} \quad \partial\Omega \end{cases}$$

Theorem (Caloz, Dauge, P., 09)

If the surface Σ is Lipschitz, under Hypothesis (SH), there exist σ_0 and $C > 0$, such that for all $\sigma \geqslant \sigma_0$, $(\mathbf{P}_{\sigma})$ with B.C. and $\mathbf{j} \in \mathrm{H}(\mathrm{div}, \Omega)$ has a unique solution $(\mathbf{E}, \mathbf{H})$ in $\mathrm{L}^2(\Omega)^2$, and

$$\|\mathbf{E}\|_{0,\Omega} + \|\mathbf{H}\|_{0,\Omega} + \sqrt{\sigma}\, \|\mathbf{E}\|_{0,\Omega_-} \leqslant C\, \|\mathbf{j}\|_{\mathrm{H}(\mathrm{div},\Omega)}$$

Application: Convergence of asymptotic expansion for large conductivity

Outline

References and Notations

Asymptotic Expansion as $\sigma \to \infty$ of solutions of $(\mathbf{P}_{\sigma})$ *when* Σ *is smooth* :

- E.P. STEPHAN, R.C. MCCAMY (83-84-85)
  Plane Interface and Eddy Current Problems
- H. HADDAR, P. JOLY, H.N. NGUYEN (08)
  Generalized Impedance Boundary Conditions
- G. CALOZ, M. DAUGE, V. PÉRON (JMAA 10)
- M. DAUGE, E. FAOU, V. PÉRON (CRAS 10)
- PÉRON (09)

Hypothesis

1. Σ *is a Smooth Surface*
2. ω *satisfies the Spectral Hypothesis (SH)*
3. $\mathbf{j}$ *is smooth and* $\mathbf{j} = 0$ *in* Ω_-

Asymptotic Expansion

$$\delta := \sqrt{\omega\varepsilon_0/\sigma} \longrightarrow 0 \quad \text{as} \quad \sigma \to \infty$$

By Theorem there exists δ_0 s.t. for all $\delta \leqslant \delta_0$, the solution $\mathbf{H}_{(\delta)}$ to $\left(\mathbf{P}_{\underline{\sigma}}\right)$:

$$\mathbf{H}^+_{(\delta)}(\mathbf{x}) \approx \mathbf{H}^+_0(\mathbf{x}) + \delta\mathbf{H}^+_1(\mathbf{x}) + \mathcal{O}(\delta^2)$$

$$\mathbf{H}^-_{(\delta)}(\mathbf{x}) \approx \mathbf{H}^-_0(\mathbf{x};\delta) + \delta\mathbf{H}^-_1(\mathbf{x};\delta) + \mathcal{O}(\delta^2)$$

$$\text{with} \quad \mathbf{H}^-_j(\mathbf{x};\delta) = \chi(y_3)\,\mathbf{V}_j(y_\beta, \frac{y_3}{\delta})\,.$$

(y_β, y_3) : "*normal coordinates* " to Σ in a tubular region $\mathcal{U}_-$ of Σ in Ω_-

$$\mathbf{H}^+_j \in \mathrm{H}(\mathrm{curl}, \Omega_+) \quad \text{and} \quad \mathbf{V}_j \in \mathrm{H}(\mathrm{curl}, \Sigma \times \mathbb{R}_+) \quad \textit{profiles}.$$

$$\|\mathbf{H}^-_j(\mathbf{x};\delta)\|_{0,\Omega_-} \leqslant C_j\sqrt{\delta} \quad \text{for all} \quad j \in \mathbb{N}$$

Profiles of the Magnetic Field

$\mathbf{V}_j =: (\mathcal{V}_j^\alpha \ ; \ v_j)$ in coordinates (y_β, Y_3) with $Y_3 = \frac{y_3}{\delta}$

$$\mathbf{V}_0(y_\beta, Y_3) = \mathbf{h}_0(y_\beta)\, \mathrm{e}^{-\lambda Y_3}$$

$$\mathcal{V}_1^\alpha(y_\beta, Y_3) = \Big[\mathrm{h}_1^\alpha + Y_3\Big(\mathcal{H}\, \mathrm{h}_0^\alpha + b_\sigma^\alpha\, \mathrm{h}_0^\sigma\Big)\Big](y_\beta)\, \mathrm{e}^{-\lambda Y_3}$$

Here,

$\mathcal{H}$ mean curvature of Σ

$$\mathbf{h}_0(y_\beta) = (\mathbf{n} \times \mathbf{H}_0^+) \times \mathbf{n}(y_\beta, 0) \quad \text{and} \quad \mathrm{h}_j^\alpha(y_\beta) := (\mathbf{H}_j^+)^\alpha(y_\beta, 0)$$

$$\lambda = \omega\sqrt{\varepsilon_0\mu_0}\, \mathrm{e}^{-i\pi/4}$$

Application

A New Definition of the Skin Depth

$$\mathbf{V}_{(\delta)}(y_\alpha, y_3) := \mathbf{H}^-_{(\delta)}(\mathbf{x}), \quad y_\alpha \in \Sigma, \quad 0 \le y_3 < h_0$$

Definition

Let Σ be a smooth surface, and $\mathbf{j}$ s.t. $\mathbf{V}_{(\delta)}(y_\alpha, 0) \neq 0$. The skin depth is the smallest length $\mathcal{L}(\sigma, y_\alpha)$ defined on Σ s.t.

$$\|\mathbf{V}_{(\delta)}(y_\alpha, \mathcal{L}(\sigma, y_\alpha))\| = \|\mathbf{V}_{(\delta)}(y_\alpha, 0)\| \, \mathrm{e}^{-1}$$

Theorem (Dauge, Faou, P., 10)

Let Σ be a regular surface with mean curvature $\mathcal{H}$, and assume $\mathbf{h}_0(y_\alpha) \neq 0$.

$$\mathcal{L}(\sigma, y_\alpha) = \ell(\sigma)\Big(1 + \mathcal{H}(y_\alpha)\,\ell(\sigma) + \mathcal{O}(\sigma^{-1})\Big), \quad \sigma \to \infty$$

Key of the proof:

$$\|\mathbf{V}_{(\delta)}\|^2 = \Big[\|\mathbf{h}_0\|^2 + 2y_3\mathcal{H}\,\|\mathbf{h}_0\|^2 + 2\delta\,\mathrm{Re}\langle \mathbf{h}_0, \mathbf{h}_1\rangle + \mathcal{O}\big((\delta + y_3)^2\big)\Big]\,\mathrm{e}^{-2y_3/\ell(\sigma)}$$

Outline

Axisymmetric domains

The meridian domain

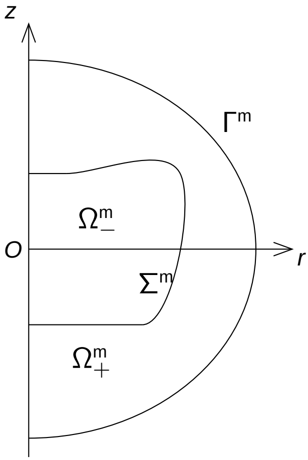


Figure: The meridian domain $\Omega^m = \Omega^m_- \cup \Omega^m_+ \cup \Sigma^m$

Reduction problem

$$\begin{cases} (\mathrm{curl}\,\mathbf{H})_r = \frac{1}{r}\partial_\theta H_z - \partial_z H_\theta \ , \\ (\mathrm{curl}\,\mathbf{H})_\theta = \partial_z H_r - \partial_r H_z \ , \\ (\mathrm{curl}\,\mathbf{H})_z = \frac{1}{r}\Big(\partial_r\big(rH_\theta\big) - \partial_\theta H_r\Big) \ . \end{cases}$$

The Maxwell problem is axisymmetric : the coefficients do not depend on the angular variable θ.

Reduction problem

$$\begin{cases} (\operatorname{curl} \mathbf{H})_r = \frac{1}{r}\partial_\theta H_z - \partial_z H_\theta \ , \\ (\operatorname{curl} \mathbf{H})_\theta = \partial_z H_r - \partial_r H_z \ , \\ (\operatorname{curl} \mathbf{H})_z = \frac{1}{r}\Big(\partial_r (rH_\theta) - \partial_\theta H_r\Big) \ . \end{cases}$$

The Maxwell problem is axisymmetric : the coefficients do not depend on the angular variable θ.

$\mathbf{H}$ is axisymmetric iff $\breve{\mathbf{H}} := \big(H_r, H_\theta, H_z\big)$ does not depend on θ.

Assume that the right-hand side is axisymmetric and orthoradial. Then, $\mathbf{H}_{(\delta)}$ is axisymmetric and orthoradial :

$$\breve{\mathbf{H}}_{(\delta)}(r,\theta,z) = (0, \mathrm{h}_{(\delta)}(r,z), 0).$$

Configurations chosen for computations

Configuration A

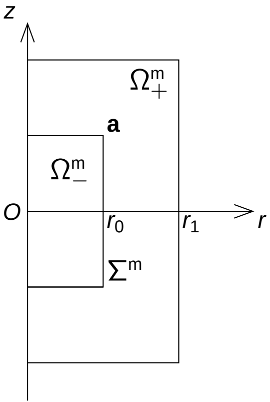


Figure: The meridian domain Ω^m in configuration A

Configurations chosen for computations

Configuration B

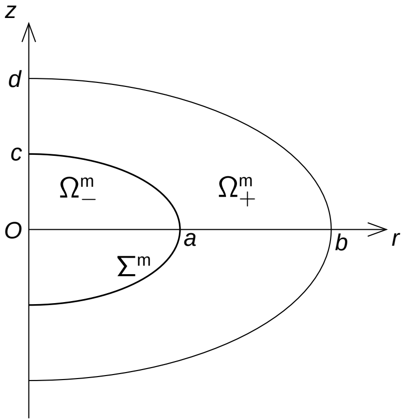


Figure: The meridian domain Ω^m in configuration B

Outline

Finite Element Method

- High order elements available in the finite element library Mélina
- $\mathrm{h}_{(\delta)}^{p,\mathfrak{M}}$: the computed solution of the discretized problem with an interpolation degree p and a mesh $\mathfrak{M}$

$$A_{\sigma}^{p,\mathfrak{M}} := \|\mathrm{h}_{(\delta)}^{p,\mathfrak{M}}\|_{\mathrm{L}_1^2(\Omega_-^{\mathrm{m}})} \quad \text{with} \quad \sigma = \omega\varepsilon_0\delta^{-2}$$

$$\|v\|^2_{\mathrm{L}_1^2(\Omega_-^{\mathrm{m}})} = \int_{\Omega_-^{\mathrm{m}}} |v|^2 \, rdrdz \; .$$

- In the computations, the angular frequency $\omega = 3.10^7$.

Interpolation degree

Configuration B

We first check the convergence when the interpolation degree p of the finite elements increases.

We consider the discretized problem with different degrees: Q_p, $p = 1, \ldots, 20$, and with 2 different meshes:

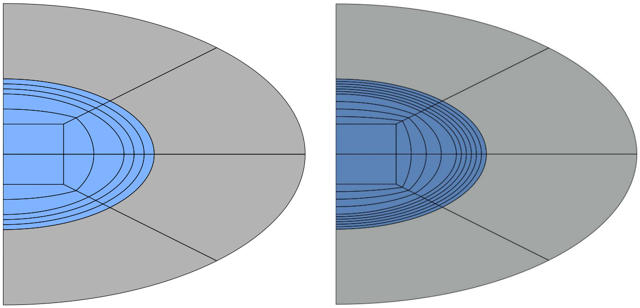
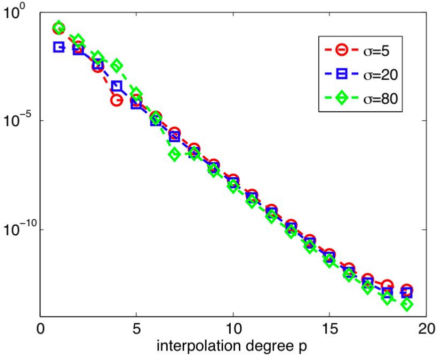


Figure: The meshes $\mathfrak{M}_3$ and $\mathfrak{M}_6$

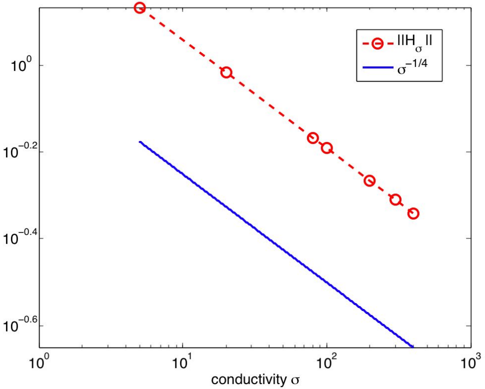
We represent the absolute value of the difference between the weighted norms $A_\sigma^{p,\mathfrak{M}_3}$ and $A_\sigma^{20,\mathfrak{M}_6}$, versus p in semilogarithmic coordinates



SCHWAB, SURI (96)

theoretical results of convergence for the p-version of problems with boundary layers

We plot in log-log coordinates the weighted norm $A_\sigma^{16,\mathfrak{M}_3}$ with respect to σ with red circles.



The figure shows that $A_\sigma^{16,\mathfrak{M}_3}$ behaves like $\sigma^{-1/4}$ (solid line) when $\sigma \longrightarrow \infty$. This behavior is consistent with the asymptotic expansion.

Configuration A

We consider a family of meshes with square elements $\mathfrak{M}_k$, with size $h = 1/k$

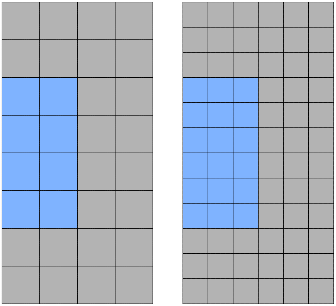
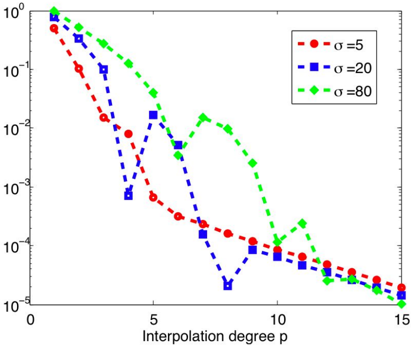


Figure: Meshes $\mathfrak{M}_2$, and $\mathfrak{M}_3$ in configuration A

Configuration A

We represent the absolute value of the difference between $A_\sigma^{p,\mathfrak{M}_2}$ and $A_\sigma^{16,\mathfrak{M}_3}$, versus p in semilogarithmic coordinates



The figure shows that $A_\sigma^{p,\mathfrak{M}_2}$ approximates $A_\sigma^{16,\mathfrak{M}_3}$ better than 10^{-4} when $p \geqslant 12$.

Outline

Skin effect in configuration B

$$\left| \operatorname{Im} \mathsf{h}^{+}_{(\delta)} \right| = \mathcal{O}(\delta) \,.$$

Thus, the imaginary part of the computed field is located in the conductor.

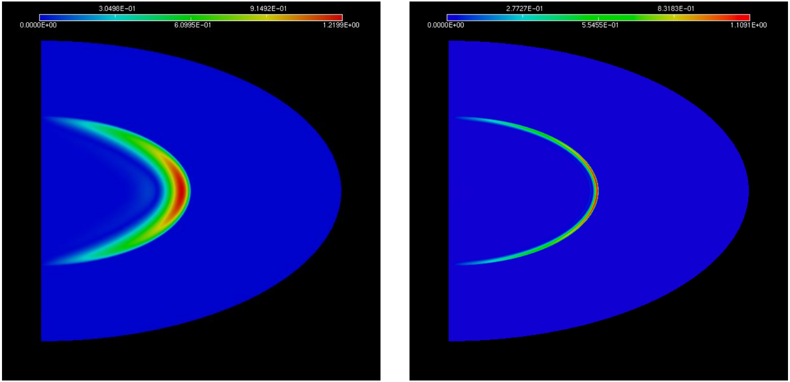


Figure: Configuration B. $\left| \operatorname{Im} \mathrm{H}_\sigma \right|$ when $\sigma = 5$ and $\sigma = 80$

Skin effect in configuration A

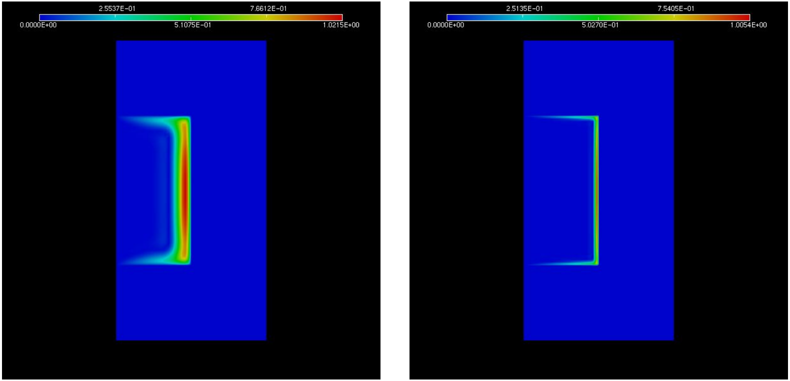


Figure: Configuration A. $|\text{Im}\,\mathrm{H}_\sigma|$ when $\sigma = 5$ and $\sigma = 80$

Influence of the Geometry on the Skin effect

Configuration B

$\mathcal{H} > 0$ on the left, and $\mathcal{H} < 0$ on the right

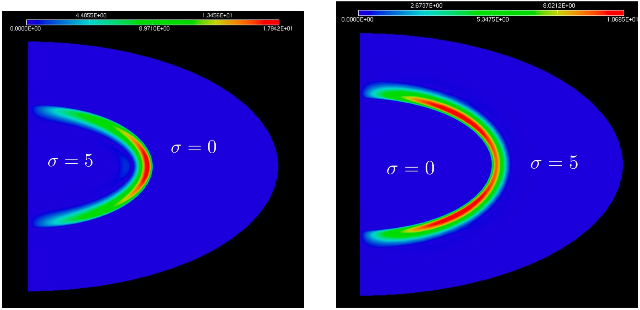


Figure: $|\operatorname{Im} \mathrm{H}_\sigma|$, $\sigma = 5$

The skin depth is larger when the mean curvature of the conducting body surface is larger.

Outline

Configuration B

We perform numerical treatments from computations in configuration B.

We extract values of $\log_{10}\left|\mathrm{H}_\sigma\right|$ in Ω^{m}_{-} along the axis $z = 0 : y_3 = 2 - r$.

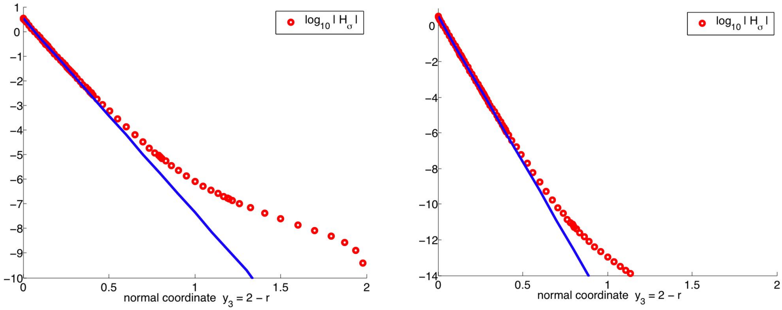
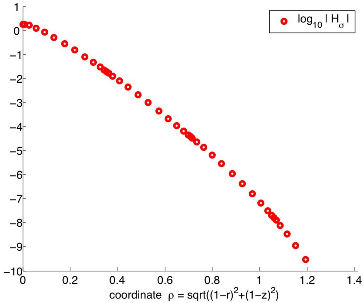


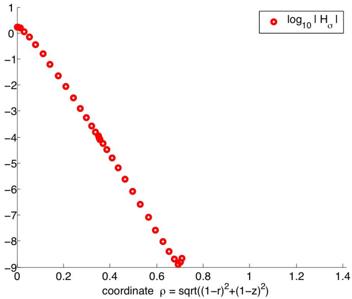
Figure: On the left $\sigma = 20$. On the right, $\sigma = 80$.

The curves exactly behave like lines: the exponential decay is obvious.

Configuration A

We extract values of $\log_{10}\left|\mathrm{H}_\sigma\right|$ in Ω^{m}_{-} along the diagonal axis $r = z$





Here, $\rho = \sqrt{(1-r)^2 + (1-z)^2}$ is the distance to the corner point $\mathbf{a}(r = 1, z = 1)$.

The exponential decay is not obvious in configuration A.

To measure a possible exponential decay, we define the slopes

$$\tilde{s}_i(\sigma) := \frac{\log_{10} |\mathrm{H}_\sigma(r_i, z_i)| - \log_{10} |\mathrm{H}_\sigma(z_{i+1}, r_{i+1})|}{\rho_{i+1} - \rho_i} .$$

Here, $\rho_i := \sqrt{(1 - r_i)^2 + (1 - z_i)^2}$ is the distance from the extraction points (r_i, z_i) to $\mathbf{a}$.

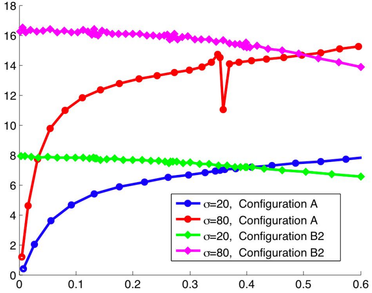


Figure: The graphs of the slopes $\tilde{s}_i(\sigma)$

Asymptotics in the conducting part

In configuration A, the slopes tend to 0, which means that there is no exponential convergence near the corner.

Nevertheless, a sort of exponential convergence is restored in a region further away from the corner.

The principal asymptotic contribution inside the conductor is a profile globally defined on a sector $\mathcal{S}$ (of opening $\frac{\pi}{2}$) solving the model Dirichlet problem

$$\begin{cases} -i(\partial_X^2 + \partial_Y^2)\mathrm{v}_0 + \kappa^2 \mathrm{v}_0 & = & 0 & \text{in} & \mathcal{S}\,, \\ \mathrm{v}_0 & = & \mathrm{h}_0^+(\mathbf{a}) & \text{on} & \partial\mathcal{S}\,, \end{cases}$$

instead the 1D problem in configuration B

$$\begin{cases} -i\partial_Y^2 \mathrm{v}_0 + \kappa^2 \mathrm{v}_0 & = & 0 & \text{for} & 0 < Y < +\infty\,, \\ \mathrm{v}_0 & = & \mathrm{h}_0^+ & \text{for} & Y = 0\,. \end{cases}$$